Triple Integrals

Idea: Integrate functions of 3 variables (Nothing really changes)

Now $\iiint_R f(x,y,z)\,dV$ has R a solid region in 3 space, w/ f defined thereon

Ex Compute $\iiint_E (xy+z^2)\,dV$ where $\underbrace{E=[0,2]\times[0,1]\times[0,3]}_{\text{Rectangular prism in } \mathbb{R}^3}$

Sol: $\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3} (xy+z^2)\,dz\,dy\,dx$ → Note: We can choose order as long a we carefully reparameterize like Fubini's Theorem

$= \int_0^2\int_0^1 \left[xyz+\frac{1}{3}(z)^3\right]_0^3 dy\,dx$

$= \int_0^2\int_0^1 (3xy+9-0)\,dy\,dx$

$= \int_0^2 \left[\frac{3}{2}xy^2+9y\right]_0^1 dx \quad = \int_0^2 \frac{3}{2}x+9-0\,dx$

$= \left[\frac{3}{4}x^2+9x\right]_0^2 \quad = \frac{3}{4}(4)+18-0 = 21$ ☒

Ex Compute $\iiint_R (2x-y)\,dV$ $R=\{(x,y,z): 0\le z\le 2,\ 0\le y\le z^2,\ 0\le x\le y-z\}$

R is parametrized w/ $\{(x,y,z): \underbrace{c_1\le z\le c_2}_{\text{constant}},\ g_1(z)\le y\le g_2(z),\ h_1(y,z)\le x\le h_2(y,z)\}$

BAD ORDER
if not reparameterized

Q: What would integral look like in dydxdz order?

$$\int_{z=0}^{2}\int_{x=0}^{y-z}\int_{y=0}^{z^2}(2x-y)\,dy\,dx\,dz$$

inner: $\int_{y=0}^{z^2}(2x-y)dy = \left[2xy-\frac{1}{2}y^2\right]_0^{z^2}$

$= 2xz^2-\frac{1}{2}z^4-0$

Middle: $\int_0^{y-z}\left(2xz^2-\frac{1}{2}x^4\right)dx = \left[x^2z^2-\frac{1}{2}xz^4\right]_0^{y-z}$

$= \underbrace{(y-z)^2z^2-\frac{1}{2}(y-z)z^4}_{y \text{ is constant now}}$

BAD!

Now do given order

Sol (in x, y, z order)

$$\int_{z=0}^{2}\int_{y=0}^{z^2}\int_{x=0}^{y-z}(2x-y)\,dx\,dy\,dz$$

inner $\int_{z=0}^{2}\int_{y=0}^{z^2}\left[x^2-xy\right]_0^{y-z}dy\,dz = \int_{z=0}^{2}\int_{y=0}^{z^2}(y-z)^2-(y-z)y\,dy\,dz$

Middle $\int_{z=0}^{2}\int_{y=0}^{z^2} z^2-yz\,dy\,dz = \int_{z=0}^{2}\left[z^2y-\frac{1}{2}y^2z\right]_0^{z^2}$

$= \int_{z=0}^{2} z^4-\frac{1}{2}z^5-0\,dz$

Outer $\int_0^2 z^4-\frac{1}{2}z^5-0\,dz = \left[\frac{1}{5}z^5-\frac{1}{12}z^6\right]_0^2 = \frac{32}{5}-\frac{64}{12}$

$= \boxed{16/15}$

Reparameterize bad order $dy\,dx\,dz$

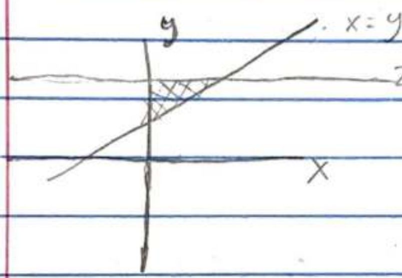
Reparameterize $\begin{cases} 0 \le z \le 2 \\ 0 \le y \le z^2 \\ 0 \le x \le y-z \end{cases}$

bounding pieces: $x=0 \quad x=y-z$

$y=0 \quad y=z^2$ — to flip, check intersections

$z=0 \quad z=2$

$\therefore$ New bounds are $R=\{(x,y,z): 0\le z\le 2,\ 0\le x\le z^2 z,\ 0\le y\le z^2\}$



$z^2 = x+z$

$\therefore x = z^2 - z$

$\therefore 0 \le x \le z^2 - z$

$\therefore \boxed{x+z \le y \le z^2}$ correct reparameterize y

Sol2: $\displaystyle\int_{z=0}^{2}\int_{x=0}^{z^2-z}\int_{y=x+z}^{z^2} (2x-y)\,dy\,dx\,dz$

inner $\displaystyle\int_{y=x+z}^{z^2} (2x-y)\,dy = \left[2xy - \tfrac{1}{2}y^2\right]_{x+z}^{z^2} = \left(2xz^2 - \tfrac12 z^4\right) - \left(2x(x+z) - \tfrac12 (x+z)^2\right)$

$= 2xz^2 - \frac12 z^4 - 2x^2 - \cancel{2}xz + \frac12 x^2 + \cancel{xz} - \frac12 z^2$

$= 2xz^2 - \frac12 z^4 - \frac32 x^2 - xz + \frac12 z^2$

Middle $\displaystyle\int_{x=0}^{z^2-z} 2xz^2 - \tfrac12 z^4 - \tfrac32 x^2 - xz + \tfrac12 z^2\,dx$

$= \left[x^2z^2 - \frac12 xz^4 - \frac12 x^3 - \frac12 x^2 z + \frac12 xz^2\right]_0^{z^2-z}$

$= (z^2-z)z^2 - \frac12 (z^2-z)z^4 - \frac12 (z^2-z)^3 - \frac12 (z^2-z)^2 z + \frac12 (z^2-z)z^2 - 0$

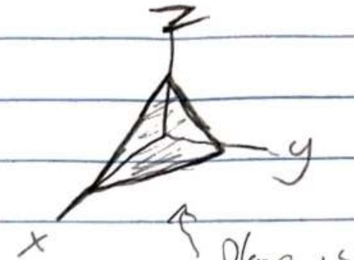
$= (z^2-z)\left((z^2-z)z^2 - \frac12 z^4 - \frac12 (z^2-z)^2 - \cancel{\tfrac12 (z^2-z)z} + \frac12 z^2\right)$

$= (z^2-z)\left(z^4 - z^3 - \frac12 z^4 - \frac12 (z^4 - 2z^3 + z^4) - \frac12 z^3 + \frac12 z^2\right)$

$= (z^2-z)\left(-\frac12 z^3 - \frac12 z\right)$

$= -\frac12 z^2 (z^3 + z - z^2 - 1)$ | Mistake Somewhere here. Email to come later

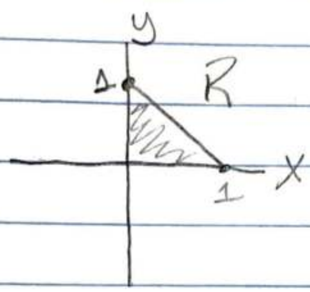
Ex Compute the volume of the tetrahedron $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$



Sol: $\mathrm{Vol}(T) = \iiint_T 1 \, dV$

Now parameterize T

First look at possible (x,y) pairs



in xy-plane: $\{(x,y) : 0 \le x \le 1, \ 0 \le y \le 1-x\}$

In T: $0 \le z \le 1-x-y$

$\therefore T = \{(x,y,z) : 0 \le x \le 1, \ 0 \le y \le 1-x, \ 0 \le z \le 1-x-y\}$

$$\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 1 \, dz\,dy\,dx$$

inner $$\int_{x=0}^{1} \int_{y=0}^{1-x} \Big[z \Big]_0^{1-x-y} dy\,dx$$

middle $$\int_{x=0}^{1} \int_{y=0}^{1-x} 1-x-y \, dy\,dx = \int_{x=0}^{1} \Big[y - xy - \tfrac{1}{2}y^2 \Big]_0^{1-x} dx$$

outer $$\int_{x=0}^{1} (1-x) - x(1-x) - \tfrac{1}{2}(1-x)^2 \, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} (1-x)^2 \, dx = -\frac{1}{2} \cdot \frac{1}{3} \Big[(1-x)^3 \Big]_0^1$$

$$= -\frac{1}{6}(0-1)$$

$$= \boxed{\frac{1}{6}}$$